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Partial confluence and closed subsets of a graph

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Abstract

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If f is a map from a continuum X onto a continuum Y, then a subcontinuum K of Y is a w_f-set if there is a subcontinuum K' of X such that $f(K') = K$. If K is a closed subset of a continuum Y, $P(K)$ is the smallest cardinal such that for each map f of a continuum onto Y, K is the union of $P(K)$ or fewer w_f-sets. If G is a graph and K is a closed subset of G with dense interior, then $P(K) = 2(\beta(G) - \beta(\mathrm{cl}(G \setminus K))) + c(\mathrm{cl}(G \setminus K)) + t(K) - 1$, where $\beta(G)$ is the first Betti number of G, $c(\mathrm{cl}(G \setminus K))$ is the number of components of $\mathrm{cl}(G \setminus K)$, and $t(K)$ is the number of terminal points of G in K.

Keywords: Continuum; Weakly confluent; Partial confluence; Graph.

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Introduction

Though this paper is presented in the fairly general language of continuum theory, there are simple concrete examples that can illustrate the results reasonably well. Consider the following situation. A network of wires running through a network of conduits can be an example of a continuous function from a graph (the wires) onto a graph (the conduits). Suppose that a building on a college campus has a system of conduits through which computer wires have been run with the intent of offering access at each point along the conduits to a campus wide computer network. Over the years the wires have been threaded through the conduits to meet the needs of the day, and records have been lost concerning the location of the wires. The wires in the building need to be replaced, but, if some

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are found to be redundant, they will be routed around the building at a substantially smaller cost. There is a blueprint of the conduits for the whole campus, and it is possible with circuit testers to disconnect the building from the campus network and count the number of components of the computer network which are in the building. If there are more than $2(\beta(G)-\beta(\mathrm{cl}(G\setminus K)))+c(\mathrm{cl}(G\setminus K))+t(K)-1$ components, where G is the graph formed by the conduits for the whole campus, K is the conduit graph for the building, and the remaining expressions are easy to compute and will be explained later, then there is redundancy in the building. That is, at least one of the components of the computer network can be rerouted around the building without affecting the availability of the computer network at each point of K. Finding the extra component is beyond the powers of our formula. Also, there may be redundancy even if the number of components is less than $2(\beta(G)-\beta(\mathrm{cl}(G\setminus K)))+c(\mathrm{cl}(G\setminus K))+t(K)-1$.

Definitions

A *continuum* is a compact connected metric space. A *map* is a continuous function. If f is a map from a continuum X onto a continuum Y, then a subcontinuum K of Y is a w_f-set if there is a subcontinuum K' of X such that $f(K')=K$. A *maximal w_f-set in* K is a w_f-set which is contained in K and which is not a proper subset of a w_f-set which is contained in K. If K is a subcontinuum of the continuum M, and f is a map from a continuum onto M, then $P_f(K)$ is the smallest integer such that K is the union of $P_f(K)$ or fewer w_f-sets, $P(K)$ is the smallest cardinal such that for each map f of a continuum onto M, K is the union of $P(K)$ or fewer w_f-sets, and $P_f(M)$ is the smallest cardinal such that, for each subcontinuum K of M, K is the union of $P_f(M)$ of fewer w_f-sets, and $P(M)$ is the smallest integer such that for each map f of a continuum onto M, and each subcontinuum K of M, K is the union of $P(M)$ or fewer w_f-sets. A map f onto a continuum M is *weakly confluent* provided $P_f(M)=1$, and f is *partially confluent* if $P_f(M)$ is finite.

The last two definitions are included to help the reader place the results of this paper into existing literature on maps between continua, to explain the mention of partially confluent maps in the title of this paper, and to indicate where the P comes from in the notation "$P(K)$".

A subcontinuum A of a continuum X is a *free arc* in X if A is an arc such that the boundary of A is contained in the set of endpoints of A. A continuum G is a *graph* if it is the union of a finite number of free arcs. An *interior arc* of a graph G is an arc which is contained in the interior of a free arc in G. For a graph G let $t(G)$ be the number of points of order one in G, called *terminal points*. Let $\beta(G)$ be the first Betti number for G. In [2] it was shown that $P(G)=3\beta(G)+t(G)-1$.

For most of this paper it will not be neccessary to specify which points of G are vertices, but, in some instances, where a subcontinuum K is specified, the vertices of G will be assumed to be all points of G with local order not equal to two together with the terminal points of K. That will make K a subgraph of G. For a subset K of the graph G, first assume that K has finitely many components, since the case K has infinitely many components is trivial, then let $e(K)$ be the number of edges of K as determined by the vertices of G which are in K and the terminal points of K, let $v(K)$ be the number of vertices and terminal points which are in K, let $c(K)$ be the number of components of K, and let $t(K)$ be the number of terminal points of G which are in K. Let $p(K) = 2(e(K) - v(\text{int}(K))) - c(\text{cl}(G\setminus K)) + t(K) + 1$. If $c(\text{cl}(G/K))$ is finite, from the Euler formula we have

$$\begin{aligned}\beta(\text{cl}(G\setminus K)) &= e(\text{cl}(G\setminus K)) - v(\text{cl}(G\setminus K)) + c(\text{cl}(G\setminus K))\\ &= (e(G) - e(K)) - (v(G) - v(\text{int}(K)))\\ &\quad + c(\text{cl}(G\setminus K))\\ &= (e(G) - v(G)) + 1 - (e(K) - v(\text{int}(K)))\\ &\quad + c(\text{cl}(G\setminus K)) - 1\\ &= \beta(G) - (e(K) - v(\text{int}(K))) + c(\text{cl}(G\setminus K)) - 1.\end{aligned}$$

From this equation if follows that $p(K) = 2(\beta(G) - \beta(\text{cl}(G\setminus K))) + c(\text{cl}(G\setminus K)) + t(K) - 1$.

The main result of this paper is that if K is a proper subset of a graph G such that $K = \text{cl}(\text{int}(K))$, then $P(K) = 2(\beta(G) - \beta(\text{cl}(G\setminus K))) + c(\text{cl}(G\setminus K)) + t(K) - 1$. To show how this formula works in one specific case, we have drawn, in Fig. 1, a graph G on the right with a subset K indicated by the thicker edges. Note that $e(K) = 7$, $v(\text{int}(K)) = 4$, $c(\text{cl}(G\setminus K)) = 3$, and $t(K) = 2$. So, $p(K) = 2(e(K) - v(\text{int}(K))) - c(\text{cl}(G\setminus K)) + t(K) + 1 = 7$. Also, note that $\beta(G) = 2$, $\beta(\text{cl}(G\setminus K)) = 0$. So, by the alternate formula, $p(K) = 2(\beta(G) - \beta(\text{cl}(G\setminus K))) + c(\text{cl}(G\setminus K)) + t(K) - 1 = 7$. Therefore, it is possible to find a map f from a continuum onto G such that K is the union of seven w_f-sets, and K is not the union of fewer than seven w_f-sets. Moreover, if g is any map from a continuum onto G, then K is the union of seven or fewer w_g-sets. For the map f indicated in Fig. 1, there are seven components of $f^{-1}(K)$ which are indicated by thicker edges. The image of each of these components contains points of K not contained in the image of another component of $f^{-1}(K)$. Thus, $P_f(K) = 7$.

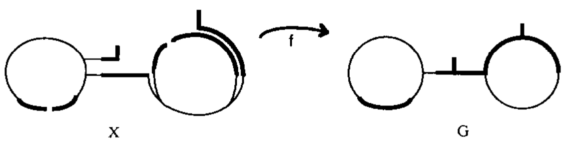

Fig. 1.

The proof of $P(K) = 2(\beta(G) - \beta(\mathrm{cl}(G \setminus K))) + c(\mathrm{cl}(G \setminus K)) + t(K) - 1$

Lemma 1. *The maximum value of $p(K)$ for all subcontinua K of the graph G is $P(G) = 3\beta(G) + t(G) - 1$.*

Proof. The graph G contains a subcontinuum K which is acyclic, and such that the interior of K contains all of the vertices and terminal points of G. For this subcontinuum K, $c(\mathrm{cl}(G \setminus K)) = \beta(G)$, $\beta(\mathrm{cl}(G \setminus K)) = 0$ and $t(K) = t(G)$. So $p(K) = 2(\beta(G) - \beta(\mathrm{cl}(G \setminus K))) + c(\mathrm{cl}(G \setminus K)) + t(K) - 1 = 3\beta(G) + t(G) - 1$.

Suppose K is any subcontinuum of the graph G. The number of components of $\mathrm{cl}(G \setminus K)$ which do not contain a terminal point of G is not greater than $\beta(G)$. The number of components of $\mathrm{cl}(G \setminus K)$ which do contain a terminal point of G is not greater than $t(G) - t(K)$. Thus, $\beta(G) + t(G) - t(K) \geqslant c(\mathrm{cl}(G \setminus K))$, and it follows that $3\beta(G) + t(G) - 1 \geqslant 2(\beta(G) - \beta(\mathrm{cl}(G \setminus K))) + c(\mathrm{cl}(G \setminus K)) + t(K) - 1$. □

The details in the proof of the following lemma can also be found in [2, Lemma 2], but the assumptions there are not quite the same, so it is repeated here.

Lemma 2. *Suppose K is a closed subset of a continuum M, and f is a map from a continuum X onto M. Suppose further that D_1 and D_2 are closed disjoint nonempty subsets of $\mathrm{cl}(M \setminus K)$ such that $\mathrm{cl}(M \setminus K) \subset D_1 \cup D_2$, and no component of $\mathrm{cl}(M \setminus K)$ intersects D_1 and D_2. Then, there is a w_f-set W in K such that $W \cap D_1 \neq \emptyset \neq W \cap D_2$.*

Proof. Let $C_1 = K \cap D_1$, and let $C_2 = K \cap D_2$. Let E be a continuum in X which is irreducible from $f^{-1}(C_1)$ to $f^{-1}(C_2)$. If $f(E) \cap (D_1 \setminus K) \neq \emptyset$, then there is a component of $E \setminus f^{-1}(D_1 \setminus K)$ which intersects $f^{-1}(C_2)$, and this component would be a proper subcontinuum of E which intersects $f^{-1}(C_2)$ and which intersects $f^{-1}(C_1)$, since $f^{-1}(C_1)$ contains the boundary in E of $E \setminus f^{-1}(D_1 \setminus K)$. So, $f(E) \cap (D_1 \setminus K) = \emptyset$, and, similarly, $f(E) \cap (D_2 \setminus K) = \emptyset$. The required w_f-set is $f(E)$. □

Lemma 3. *If K is a proper subcontinuum of the graph G, then $P(K) \leqslant p(K)$.*

Proof. Let K be a subcontinuum of the graph G. We will prove the theorem by induction on $P(G) - p(K)$.

According to Lemma 1, the minimum value of $P(G) - p(K)$ is 0, and if $P(G) - p(K) = 0$ then $P(K) \leqslant P(G) = p(K)$.

Suppose the theorem is true for subcontinua K of G such that $P(G) - p(K) = i$. Suppose K is a subcontinuum of G and $P(G) - p(K) = i + 1$. Suppose f is a map from a continuum onto G. We must show that K is contained in the union of $p(K)$ of fewer w_f-sets in K. It follows from Lemma 2 that there is a collection of maximal w_f-sets in K, $\mathscr{W} = \{W_1, W_2, \ldots, W_\alpha\}$, with $\alpha \leqslant c(\mathrm{cl}(G \setminus K)) - 1 + t(K)$

such that $\mathrm{cl}(G\setminus K)\cup(\bigcup\mathscr{W})$ is connected, and each terminal point of G which is contained in K is contained in an element of $\mathscr{W}$. If $\mathscr{W}$ covers K, then, since $c(\mathrm{cl}(G\setminus K))-1+t(K)\leqslant p(K)$, we are done.

If $\mathscr{W}$ does not cover K, then there is a collection of maximal w_f-sets in K, $\mathscr{W}'=\{W_1', W_2', \ldots, W_{\alpha'}'\}$, such that, for each $0\leqslant i\leqslant\alpha'$, W_i' contains points which are not contained in any other element of $\mathscr{W}$ of $\mathscr{W}'$. We will construct a graph G' and a map f' from G' onto G such that $P_f(K)\leqslant P_{f'}(K)$. To construct the graph G', start with a copy of $\mathrm{cl}(G\setminus K)$, then take a copy of an element W_i of $\mathscr{W}$. Let b be a point in $W_i\cap\mathrm{cl}(G\setminus K)$, and let E_b be an edge of $\mathrm{cl}(G\mid K)$ which also contains b. Attach one endpoint of an arc A_b to a point in the interior of E_b in the copy of $\mathrm{cl}(G\setminus K)$ and attach the other endpoint of the arc to b in the copy of W_i. Do the same for each point b in $W_i\cap\mathrm{cl}(G\setminus K)$. Repeat this construction for each element of $\mathscr{W}$ and each element of $\mathscr{W}'$. Let f' be the map from G' onto G which sends the copy of $\mathrm{cl}(G\setminus K)$ in G' to $\mathrm{cl}(G\setminus K)$ in G, the copies of the elements of $\mathscr{W}$ and $\mathscr{W}'$ in G' to the elements of $\mathscr{W}$ and $\mathscr{W}'$ in G, and sends the arcs A_b one-to-one into E_b. The maximal $w_{f'}$-sets in K are the elements of $\mathscr{W}$ and $\mathscr{W}'$. It follows that $P_f(K)\leqslant P_{f'}(K)$, since the $w_{f'}$-sets in K are all contained in w_f-sets in K.

For each $0\leqslant i\leqslant\alpha'$ there is an interior arc A in W_i' which does not intersect an element of $\mathscr{W}$ or $\mathscr{W}'$ other than W_i'. If A separates K, then one of the two components of $\mathrm{cl}(K\setminus A)$, call it D, does not intersect $\mathrm{cl}(G\setminus K)$. For if both of the components of $\mathrm{cl}(K\setminus A)$ intersect $\mathrm{cl}(G\setminus K)$, then A would be contained in an element of $\mathscr{W}$. Now, D does not contain a terminal point of G. For if it did, A would be contained in an element of $\mathscr{W}$, since all maximal w_f-sets in K must intersect the boundary of K. In addition, no element of $\mathscr{W}$ or $\mathscr{W}'$ other than W_i' intersects D for the same reason. It follows that D, and therefore W_i', contains an interior arc which does not separate K, and which does not intersect an element of $\mathscr{W}$ or $\mathscr{W}'$ other than W_i'.

Suppose A is such an interior arc in W_i'. We will consider two cases. In the first case, A separates W_i'. Let A' be an arc contained in the interior of A. Like A, A' separates W_i' but does not separate K. Also,

$$\begin{aligned}p(\mathrm{cl}(K\setminus A')) &= 2[\beta(G)-\beta(\mathrm{cl}(G\setminus\mathrm{cl}(K\setminus A')))]\\&\quad+c(\mathrm{cl}(G\setminus\mathrm{cl}(K\setminus A')))+t(K)-1\\&=2[\beta(G)-\beta(\mathrm{cl}(G\setminus K))]\\&\quad+c(\mathrm{cl}(G\setminus K))+1+t(K)-1\\&=p(K)+1.\end{aligned}$$

Thus, $P(G)-p(\mathrm{cl}(K\setminus A'))=i$, and by the inductive hypothesis, $P(\mathrm{cl}(K\setminus A'))\leqslant p(\mathrm{cl}(K\setminus A'))=p(K)+1$. Therefore, $\mathrm{cl}(K\setminus A')$ is contained in the union of $p(K)+1$ or fewer $w_{f'}$-sets. Two $w_{f'}$-sets sets that are required to be in such a cover of $\mathrm{cl}(K\setminus A')$ are the two components of $W_i'\setminus A'$, because each component contains points of $A\setminus A'$ which are not contained in any other $w_{f'}$-set in $K\setminus A'$. On the other hand, those two components of $W_i'\setminus A'$ together with A' are contained in the single $w_{f'}$-set in K, W_i'. So, it follows that K is covered by $p(K)$ or fewer $w_{f'}$-sets in K, and, therefore, K is covered by $p(K)$ or fewer w_f-sets in K.

In the second case, A does not separate W_i'. Let x be a point in the interior of A, and let a and b be the endpoints of A. Let $D=(W_i'\setminus A)\cup[a, x]$, and let $E=(W_i'\setminus A)\cup[x, b]$. Let $\mathscr{W}''=(\mathscr{W}'\setminus\{W_i'\})\cup\{D, E\}$. Let A' be an arc in (a, x). Then, as above, $p(\mathrm{cl}(K\setminus A'))=p(K)+1$, so $P(\mathrm{cl}(K\setminus A'))\leqslant p(\mathrm{cl}(K\setminus A'))=p(K)+1$. Construct G'' just like G', but replace W_i' with D and E. Define the map f'' from G'' onto G just as f' was defined. Once again the $w_{f''}$-sets in K are all contained in w_f-sets in K, so $P_f(K)\leqslant P_{f''}(K)$. But, $\mathrm{cl}(K\setminus A')$ is contained in the union of $p(K)+1$ or fewer $w_{f''}$-sets in $K\setminus A'$. Two $w_{f''}$-sets which are required in such a cover are the two components of $\mathrm{cl}(D\setminus A')$, because each contains points of $(a, x)\setminus A'$ that are not contained in any other $w_{f''}$-set. On the other hand, those two components of $D\setminus A'$ together with A' are contained in the single $w_{f''}$-set in K, D. It follows that K is covered by $p(K)$ or fewer $w_{f''}$-sets in K, and, therefore, K is covered by $p(K)$ or fewer w_f-sets in K. □

Lemma 4. *Suppose K is a subset of the continuum M, and h is a map from M onto M' such that $h(M\setminus K)\subset h(M)\setminus h(K)$, and such that $h(K)$ contains a subset D such that $K=\mathrm{cl}(h^{-2}(D))$, and h is one-to-one on $h^{-1}(D)$. Then, if f is a map from a continuum onto M and, for some integer n, $h(K)$ is covered by n $W_{h\circ f}$-sets in $h(K)$, then K is covered by n or fewer w_f-sets in K. Thus, $P(K)\leqslant P(h(K))$.*

Proof. Suppose f is a map from a continuum X onto M. By assumption, there is a collection $\{W_1,\ldots,W_n\}$ of $w_{h\circ f}$-sets in $h(K)$ whose union is $h(K)$. There are continua $\{C_1,\ldots,C_n\}$ in X such that $h(f(C_i))=W_i$ for $1\leqslant i\leqslant n$. Then, $\{f(C_1),\ldots,f(C_n)\}$ is a collection of w_f-sets in K which covers $h^{-1}(D)$, and so the collection covers $K=\mathrm{cl}(h^{-1}(D))$. □

Lemma 5. *If K is a proper subset of the graph G such that $K=\mathrm{cl}(\mathrm{int}(K))$, then $P(K)\leqslant p(K)$.*

Proof. If $p(K)$ is countably infinite, then $c(\mathrm{cl}(G\setminus K))$ is countably infinite, and therefore, $c(K)$ is countably infinite. Each component of K is contained in the union of a finite number of w_f-sets, and, therefore, $P(K)$ is countably infinite.

Suppose $p(K)$ is finite. Then, $c(\mathrm{cl}(G\setminus K))$ must be finite, and, therefore, K has a finite number of components. In this case there is a finite collection of arcs $\mathscr{A}$ which are contained in $\mathrm{cl}(G\setminus K)$ and such that $K\cup(\bigcup\mathscr{A})$ is connected. Let h be a map from the graph G onto a graph G' which identifies the endpoints of the arcs in $\mathscr{A}$, and is the identity elsewhere. With $D=\mathrm{int}(h(K))$, h satisfies the conditions of Lemma 4. Now, $\beta(h(G))-\beta(\mathrm{cl}(h(G)\setminus h(K)))=\beta(G)-\beta(\mathrm{cl}(G\setminus K))$ and $c(\mathrm{cl}(h(G)\setminus h(K)))=c(\mathrm{cl}(G\setminus K))$, so $p(h(K))=p(K)$. According to Lemma 3, $P(h(K))\leqslant p(h(K))$. According to Lemma 4, $P(K)\leqslant P(h(K))$. Therefore, $P(K)\leqslant P(h(K))\leqslant p(h(K))=p(K)$. □

Lemma 6. *If K is a proper subset of a graph G such that $K = \mathrm{cl}(\mathrm{int}(K))$, then there is a map f from a graph onto G such that $P_f(K) = p(K)$.*

Proof. The existence of the map f will be demonstrated by induction on the number of edges in K.

Suppose K has one edge. If K separates G, or if K contains a terminal point of G, then $p(K) = 2(e(K) - v(\mathrm{int}(K))) - c(\mathrm{cl}(G\setminus K)) + t(K) + 1 = 1$. If f is the identity map of G onto G, then $P_f(K) = p(K) = 1$.

Suppose K does not separate G, and K does not contain a terminal point of G. Then, $p(K) = 2(e(K) - v(\mathrm{int}(K))) - c(\mathrm{cl}(G\setminus K)) + t(K) + 1 = 2$. The domain of the map f is $G\setminus K$ with two arcs attached where the endpoints of K used to be. The map f is the identity on $G\setminus K$ and sends each of the arcs onto one half of K. Then, $P_f(K) = p(K) = 2$.

Suppose that, for each closed subset K of G which has n edges, there is a map f from a graph onto G such that $P_f(K) = p(K)$. Suppose K is a closed subset of G with $n+1$ edges. Let E be an edge of K which intersects $\mathrm{cl}(G\setminus K)$. We will consider three cases.

In the first case, $c(\mathrm{cl}(G\setminus K)) = c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))$, and E does not contain a terminal point of G. In this case the component of $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)) = \mathrm{cl}(G\setminus K)\cup E$ which contains E is not separated by E. If both endpoints of E are in $\mathrm{cl}(G\setminus K)$, or if E is a simple closed curve, $\beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))) = \beta(\mathrm{cl}(G\setminus K)) + 1$ and $p(\mathrm{cl}(K\setminus E)) = 2(\beta(G) - \beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))) + c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))) + t(\mathrm{cl}(K\setminus E)) - 1 = p(K) - 2$. By the inductive hypothesis, there is a map f from a continuum onto G such that $P_f(\mathrm{cl}(K\setminus E)) = p(K) - 2$. We will construct a new graph G' and a map f' from this graph onto G such that $P_{f'}(K) = p(K)$. It follows from Lemma 2 that there is a collection $\mathscr{W} = \{W_1, W_2, \ldots, W_\alpha\}$ of maximal w_f-sets in $\mathrm{cl}(K\setminus E)$ which covers $\mathrm{cl}(K\setminus E)$, and such that, if D_1 and D_2 are disjoint nonempty subsets of $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))$ such that $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)) \subset D_1 \cup D_2$, and no component of $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))$ intersects D_1 and D_2, then there is an element W_i of $\mathscr{W}$ such that $W_i \cap D_1 \neq \emptyset \neq W_i \cap D_2$. Note that if D_1 and D_2 are disjoint nonempty subsets of $\mathrm{cl}(G\setminus K)$ (instead of $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))$) such that $\mathrm{cl}(G\setminus K) \subset D \cup D_2$, and no component of $\mathrm{cl}(G\setminus K)$ intersects D_1 and D_2, then there still is an element W_i of $\mathscr{W}$ such that $W_i \cap D_1 \neq \emptyset \neq W_i \cap D_2$. Let $\{b_{1,i}, b_{2,i}, \ldots, b_{\gamma,i}\}$ be the points of $W_i \cap \mathrm{cl}(G\setminus K)$, and, for each j, let E_j be an edge of $\mathrm{cl}(G\setminus K)$ which contains $b_{j,i}$. To build G', take a copy of $\mathrm{cl}(G\setminus K)$ and copies of each element of $\mathscr{W}$, and attach an arc from each $b_{j,i}$ in the copy of W_i to a point in the interior of E_j in the copy of $\mathrm{cl}(G\setminus K)$. If $a = b$ and $E_a = E_b$ take A_a and A_b disjoint except for a common point in E_a. Attach one endpoint of an arc A_a at a point in the interior of E_a in the copy of $\mathrm{cl}(G\setminus K)$ and attach one endpoint of an arc A_b at a point in the interior of E_b in the copy of $\mathrm{cl}(G\setminus K)$. This completes the construction of the graph G'. To define a map f' from G' onto G, send the copy of $\mathrm{cl}(G\setminus K)$ in G' onto $\mathrm{cl}(G\setminus K)$ in G. Send the copies of the elements of $\mathscr{W}$ in G' onto the corresponding elements of $\mathscr{W}$ in G. Send the arcs from E_i to $b_{j,i}$ in G' one-to-one into E_i in G. Send half of the arc A_a in G' along the edge E_a in G and send the other half of A_a halfway across E in G. Finally, send half of the arc A_b in G' along the edge E_b in G and send the other half of A_a halfway across E in G, mapping A_b onto the half of E not mapped onto by A_a. The number of $w_{f'}$-sets required to cover $K\setminus E$ is the same as the number of w_f-sets required to cover $K\setminus E$, which is $p(K)-2$. Two more $w_{f'}$-sets are required to cover all of K. So, $P_{f'}(K)=p(K)$.

If E is not a simple closed curve, and only one endpoint of E is in $\mathrm{cl}(G\setminus K)$, then $p(K\setminus E)=p(K)$. By induction let f be a map from a continuum onto G such that $P_f(K\setminus E)=p(K)$. Let b be the endpoint of E which is not in $\mathrm{cl}(G\setminus K)$. Construct the graph G' and the map f' in much the same way as above except that the arcs which are attached to copies of the elements of $\mathscr{W}$ at b will have their other end attached to a point on E_a instead of E_b (there is no edge E_b of $\mathrm{cl}(G\setminus K)$ which contains b), and the map will send these arcs onto E and part of E_a. (Leave off the two arcs A_a and A_b). The number of $w_{f'}$-sets required to cover K is the same as the number of w_f-sets required to cover $K\setminus E$. So, $P_{f'}(K)=p(K)$.

In the second case, $c(\mathrm{cl}(G\setminus K))=c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))+1$. In this case, E separates the component of $\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))=\mathrm{cl}(G\setminus K)\cup E$ which contains E. Therefore, $\beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))=\beta(\mathrm{cl}(G\setminus K))$, and $p(\mathrm{cl}(K\setminus E))=2(\beta(G)-\beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))))+c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))+t(\mathrm{cl}(K\setminus E))-1=p(K)-1$. Construct the graph G' in much the same way as above, but, instead of the two arcs A_a and A_b, attach a single arc to an interior point in E_a and to an interior point in E_b. Map this arc one-to-one onto part of E_a, all of E, and part of E_b. The resulting map requires one more $w_{f'}$-set to cover K than the number of w_f-sets required to cover $\mathrm{cl}(K\setminus E)$.

In the third and final case, E contains a terminal point of G. Therefore, $c(\mathrm{cl}(G\setminus K))=c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))$, $\beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))=\beta(G\setminus K)$, and $t(K\setminus E)=t(K)-1$. So, $p(\mathrm{cl}(K\setminus E))=2(\beta(G)-\beta(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E))))+c(\mathrm{cl}(G\setminus \mathrm{cl}(K\setminus E)))+t(\mathrm{cl}(K\setminus E))-1=p(K)-1$. Once again, construct G' as above, attaching a single arc to the copy of $\mathrm{cl}(G\setminus K)$ to be mapped onto E. Define f' as above, and, once again, one more w_f'-set is required to cover K than the number of w_f-sets required to cover $\mathrm{cl}(K\setminus E)$. □

The main result follows immediately from Lemmas 5 and 6.

Theorem 7. *If K is a proper subset of a graph G such that $K=\mathrm{cl}(\mathrm{int}(K))$, then $P(K)=2(\beta(G)-\beta(\mathrm{cl}(G\setminus K)))+c(\mathrm{cl}(G\setminus K))+t(K)-1$.*

If K is a closed subset of G which does contain isolated points, then $P(K)=P(\mathrm{cl}(\mathrm{int}(K)))+$ (the number of isolated points of K).

Theorem 7 is a result about maps between graphs, but its usefulness to more general types of continua is due, in part, to the fact that every one dimension metric continuum is homeomorphic to an inverse limit of graphs, and the following theorem which was not stated as generally, but was proven in [2, Theorem 3].

Theorem 8. *Suppose n is a positive integer, and the continuum $X = \varprojlim(X_\alpha, f_\alpha)$, where, for each α, X_α is a continuum, and π_α is the projection of X onto X_α. If K is a subcontinuum of X, and there is an integer n which is the smallest integer such that $P(\pi_\alpha(K)) = n$ for infinitely many α, then $P(K) \leqslant n$.*

$P(K)$ and monotone maps

The next theorem generalizes the result in [1, Theorem 3.9, p. 204], and the proof is almost identical to theirs.

Theorem 9. *If f is a monotone map from a continuum X onto a continuum Y and K is a closed subset of X, then $P(K) \geqslant P(f(K))$.*

Proof. Let g be a map from a continuum Z onto Y and let $E = \{(x, z) \mid f(x) = g(z)\}$, let π_x and π_z be the projection of E onto X and Z respectively.

$$\begin{tikzcd} E \arrow[r, "\pi_z"] \arrow[d, "\pi_x"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If $z \in Z$, then $\pi_z^{-1}(z) = f^{-1}(g(z)) \times \{z\}$. So, π_z is monotone, and, therefore, E is a continuum. There is a collection $\{W_1, W_2, \ldots, W_\alpha\}$ of continua in E such that $\alpha \leqslant P(K)$ and $\bigcup\{\pi_x(W_i)\} = K$. It follows that $\bigcup\{g(\pi_z(W_i))\} = f(K)$. Thus, $P(K) \geqslant P(f(K))$. □

This theorem is not true for open maps because a solonoid can be projected onto a circle with a confluent map f, and for any proper subcontinuum K which does not project onto all of the circle, $P(K) = 1$ and $P(f(K)) = 2$.

Combining Theorem 9 with Lemma 4, we get the following.

Theorem 10. *If f is a monotone map from a continuum M onto a continuum Y, K is a subcontinuum of Y, and f is one-to-one on a dense subset of $f^{-1}(K)$, then $P(f^{-1}(K)) = P(K)$.*

Constant $P(K)$

In this section we will consider continua M such that there is an integer n such that for each proper nondegenerate subcontinuum K of M, $P(K) = n$. An important and widely studied class of continua are those for which $n = 1$. It is called class[W].

Theorem 11. *If G is a graph such that there is an integer n such that for each proper nondegenerate subcontinuum K of G, $P(K)=n$, then G is either an arc, and $n=1$, or G is a simple closed curve, and $n=2$.*

Proof. It will suffice to show that no point of G has local order greater than two.

Suppose x is a point of G with local order greater than two, and K is a subcontinuum of G which contains x in its interior, and contains no other vertices or terminal points of G.

Note that $P(K)=2(e(K)-v(\text{int}(K)))-c(\text{cl}(G\setminus K))+t(K)+1\geqslant 2(e(K)-1)-e(K)+1\geqslant e(K)-1\geqslant 2$. If E is an edge of K, then $P(E)$ is either one or two. So $P(E)=P(K)=2$. If an edge E of K were to separate G then $P(E)=1$. Therefore, no edge of K separates G, and each component of $\text{cl}(G/K)$ intersects at least two edges of K. Thus, $P(K)=2(e(K)-v(\text{int}(K)))-c(\text{cl}(G\setminus K))+t(K)+1\geqslant 2(e(K)-1)-e(K)/2+1=3e(K)/2-1\geqslant 7/2$, a contradiction. □

An n-od is a continuum θ which contains a subcontinuum C such that $\theta\setminus C$ has at least n components. Vought has shown in [3, Theorem 1, p. 549] that if Y is a continuum for which there is an integer $n\geqslant 3$ such that Y has no n-ods, then Y does not have a subcontinuum K such that $P(K)=1$ if and only if Y is a graph in which each point is contained in a simple closed curve. From this result and Theorem 11, the following is immediate.

Theorem 12. *If Y is a continuum for which there is an integer $n\geqslant 3$ such Y has no n-ods, then there is an integer $m\geqslant 2$ such that for each proper nondegenerate subcontinuum K of Y, $P(K)=m$ if and only if Y is a simple closed curve and $m=2$.*

Question 13. Is a simple closed curve the only continuum Y such that for each proper nondegenerate subcontinuum K of Y, $P(K)=2$?

Question 14. Is a simple closed curve the only continuum Y such that there is an integer $n\geqslant 2$ such that for each proper nondegenerate subcontinuum K of Y, $P(K)=n$?

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